Calculus I Notes

Chapter Four

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Section 4.1: Maximum and Minimum Values

Absolute Maxima

Definition

A function f has an *absolute maximum* at c if

$$f(c) \geq f(x),$$

for all x in the domain of f. The number $f(c)$ is the *maximum value* of f.

An absolute maximum is also called a *global maximum*.

Absolute Minima

Definition

A function f has an *absolute minimum* at c if

$$f(c) \leq f(x),$$

for all x in the domain of f. The number $f(c)$ is the *minimum value* of f.

An absolute minimum is also called a *global minimum*.

The maximum and minimum values of f are called *extreme values*.

Local Maxima

Definition

A function f has a *local maximum* at c if

$$f(c) \geq f(x),$$

for values of x near c (that is, for all x in some interval containing c).

A local maximum can not occur at an endpoint of the domain of f.

Local Minima

Definition

A function f has a *local minimum* at c if

$$f(c) \leq f(x),$$

for values of x near c (that is, for all x in some interval containing c).

A local minimum can not occur at an endpoint of the domain of f.

Critical Numbers

Definition

A *critical number* of f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem. If f has a local maximum or minimum at c, then c is a critical number of f.

Warning! If c is a critical number of f, f does not necessarily have a local maximum or minimum at c.

Do Extreme Values Exist?

Some functions don't have an absolute maximum or minimum.

The Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum at some point in $[a, b]$, and it has an absolute minimum at some point in $[a, b]$.

The Closed Interval Method

Absolute maxima and minima can only occur at

- critical numbers or
- endpoints.

The Closed Interval Method. To find the absolute maximum and minimum values of a continuous function f on a closed interval:

1. Find the critical numbers of f.
2. Find the values of f at its critical numbers and the endpoints of the interval.
3. The largest value from Step 2 is the absolute maximum value, and the smallest of these values is the absolute minimum value.

Section 4.2: The Mean Value Theorem

Rolle's Theorem

Rolle's Theorem. Let f be a function such that

1. f is continuous on the closed interval $[a, b]$,
2. f is differentiable on the open interval (a, b), and
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem

The Mean Value Theorem. Let f be a function such that

1. f is continuous on the closed interval $[a, b]$, and
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Section 4.3: How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

First Derivative Test. Suppose that c is a critical number of a continuous function f

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, then f does not have a local maximum or minimum at c.

Concavity

Definition

Let f be a function defined on an interval I.

- If the secant line connecting $(a, f(a))$ to $(b, f(b))$ lies above the graph of f for any a and b in the interval I, then f is *concave upward* on I.
- If the secant line connecting $(a, f(a))$ to $(b, f(b))$ lies below the graph of f for any a and b in the interval I, then f is *concave downward* on I.

Concavity Test.

- If $f''(x) > 0$ for all x in I, then f is concave upward on I.
- If $f''(x) < 0$ for all x in I, then f is concave downward on I.

Inflection Points

Definition

A point P on the graph of f is an *inflection point* if f is continuous there and the graph changes concavity there (from concave up to concave down or vice versa).

Inflection points occur where f'' changes sign.

The Second Derivative Test. Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Section 4.4: Limits at Infinity; Horizontal Asymptotes

Limits at Infinity for Rational Functions

Suppose $P(x)$ and $Q(x)$ are polynomials. To find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)},$$

- Multiply the numerator and denominator by $\frac{1}{x^n}$, where n is the largest exponent that appears in the denominator.
- Evaluate the limit.

Limits at Infinity for Rational Functions

- If the numerator has smaller degree than the denominator, the limit is zero.
- If the numerator and denominator have the same degree, the limit is the ratio of the leading coefficients.
- If the numerator has larger degree than the denominator, the limit is either ∞ or $-\infty$, depending on the signs involved.

Section 4.7: Optimization Problems

Fertilizer

Problem. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil is given below.

$$Y = \frac{10N}{25 + N^2}$$

What nitrogen level gives the best yield? What is the best possible yield?

Problem. A farmer encloses a rectangular piece of land with 96 meters of fence. How long and how wide should the rectangle be to maximize the area of the enclosed land? What is the maximum possible area?

Open Top Box

Problem. If 19,200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box. Find the dimensions of the box with maximum volume.

Problem. A theater holds 1000 people. If theater tickets cost \$10, then 750 people will attend a play. For each \$0.50 reduction in price, 6 additional people will attend. Find the ticket price that maximizes revenue. How many tickets are sold at this price? What is the maximum possible revenue?

Problem. Find the point on the line $5x + y = 7$ that is closest to the point $(-4, 2)$.

Problem. Find the area of the largest rectangle that can be inscribed in the ellipse below.

$$\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$$