Calculus I Notes

Chapter Three

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Section 3.1: Derivatives and Rates of Change

Definition of Derivative

Definition

If f is a function defined near the number a, the *derivative* of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if the limit exists. If the limit does not exist, f is not *differentiable* at a, and $f'(a)$ is undefined.

The derivative can also be calculated with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Different Interpretations of the Derivative

If f is a function, the following are all the same:

- the derivative of f at a, $f'(a)$
- the slope of the tangent line to the graph of f when $x = a$,
- the instantaneous rate of change of f when $x = a$

Instantaneous Velocity

If $f(t)$ is the position (also called displacement) of an object at time t, then $f'(a)$ is the instantaneous velocity of the object when $t = a$.

Example Velocity Problems

Problem: The position of a particle at time t is $x(t) = 4t^2 + 2$. Find the instantaneous velocity of the particle when $t = 3$.

Problem: Let $s(t) = -5t^3 - 6$ be the displacement, measured in meters, of a particle at time t, measured in seconds. Find the instantaneous velocity of the particle when $t = 4$.

Displacement and position are the same thing!

Instantaneous Rate of Change

Assume y and x are physical quantities such that $y = f(x)$.

- Then $f'(a)$ is the instantaneous rate of change of y with respect to x when $x = a$.
- If x is "close" to a,

$$f'(a) \approx \frac{\Delta y}{\Delta x}.$$

- If x is "close" to a,

$$\Delta y \approx f'(a)\Delta x.$$

- Near $x = a$, a small change in x results in a corresponding change in y that is about $f'(a)$ times as big.

Units

- When x is close to a,

$$f'(a) \approx \frac{\Delta y}{\Delta x}.$$

- $$\text{Units of } f'(a) = \frac{\text{Units of } y}{\text{Units of } x}.$$

Section 3.2: The Derivative as a Function

Differentiability

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

- If the limit exists, the function is *differentiable* at x.
- If the limit does not exist, the function is *not differentiable* at x, and $f'(x)$ is not defined.

Theorem: If f is differentiable at a, then f is continuous at a.

- **Contrapositive:** If f is not continuous at a, then f is not differentiable at a.
- Contrapositives are always true.
- **Converse:** If f is continuous at a, then f is differentiable at a.
- Converses can be true or FALSE, and in this case, it is FALSE.

Three Ways for a Function Not to Be Differentiable

1. Vertical tangent line
2. Corner
3. Discontinuity

Higher Order Derivatives

Definition

The second derivative of f is the derivative of the derivative of f.

$$f''(x) = (f')'(x).$$

- The third derivative of f is the derivative of the second derivative of f.
$$f'''(x) = (f'')'(x).$$
- The 10th derivative of f is the derivative of the 9th derivative, and so on.

$$f^{(10)}(x) = (f^{(9)})'(x).$$

Increasing/Decreasing and Peaks/Troughs

- A function f is **increasing** on an interval if and only if $f'(x) \geq 0$ on that interval.
- A function f is **decreasing** on an interval if and only if $f'(x) \leq 0$ on that interval.
- The derivative of a linear function is a constant.
- If a function f has a peak or trough at x, and $f'(x)$ is defined, then $f'(x) = 0$.

Other Notation

- $$\frac{d}{dx}f(x) = f'(x)$$
- If $y = f(x)$, then
$$\frac{dy}{dx} = f'(x)$$
- $'$ and $\frac{d}{dx}$ both stand for derivative!

Section 3.3: Differentiation Formulas

The Power Rule. If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Sum, Difference, and Constant Multiple Rules

Let f and g be functions, and let c be a constant.

- **The Sum Rule.**
$$(f+g)' = f' + g'$$
- **The Difference Rule.**
$$(f-g)' = f' - g'$$
- **The Constant Multiple Rule.**
$$(cf)' = cf'$$
- Assume f, g, and h are functions, and a, b, and c are constants. Then
$$(af + bg - ch)' = af' + bg' - ch'.$$

The Product and Quotient Rules

Let f and g be functions.

- **The Product Rule.**

$$(fg)' = f'g + fg'$$

- **The Quotient Rule.**

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

- Quotient rule mnemonic.

$$\left(\frac{\text{high}}{\text{low}}\right)' = \frac{\text{low (dee high)} \; - \; \text{high (dee low)}}{\text{over the square of what's below}}$$

Section 3.4: Derivatives of Trigonometric Functions

Sine and Cosine

- $$\frac{d}{dx}\sin(x) = \cos(x).$$
- $$\frac{d}{dx}\cos(x) = -\sin(x).$$

Tangent

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Problem. A spotlight is shining on a skyscraper, and it is located on the ground 100 meters West of the skyscraper. Assume the light is shining on a part of the building that is y meters above the ground, and let θ be the angle between the beam of light and the ground. Find the rate of change of y with respect to θ when $\theta = \frac{\pi}{3}$.

Position, Velocity, and Acceleration

If the position of a particle at time t is $x(t)$, then

- the velocity is $v(t) = x'(t)$, and
- the acceleration is $a(t) = v'(t) = x''(t)$.

Problem. The position of a mass vibrating on a spring is $x(t) = 8\sin(t)$, where t is in seconds and x is in centimeters.

1. Find the velocity and acceleration at time t.
2. Find the position, velocity, and acceleration at time $t = \frac{\pi}{4}$.
3. Describe the position, velocity, and acceleration when $t = \frac{\pi}{4}$.
4. Repeat for $t = \frac{\pi}{2}$, $\frac{5\pi}{6}$, π, and $\frac{4\pi}{3}$.

Secant, Cosecant, and Cotangent

- $$\frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$
- $$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x).$$
- $$\frac{d}{dx}\cot(x) = -\csc^2(x).$$

Section 3.5: The Chain Rule

The Chain Rule

If f and g are functions such that g is differentiable at x, and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x, and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

In Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Section 3.6: Implicit Differentiation

Section 3.7: Rates of Change in the Natural and Social Sciences

Motion

- $s(t) =$ the position of a particle at time t.
- $v(t) = s'(t) =$ the velocity of the particle at time t.
- $a(t) = s''(t) =$ the acceleration of the particle at time t.

Linear Density

- M is the mass function of a rod. $M(x)$ is the mass of the part of the rod between the left endpoint and the point x units from the left endpoint.
- $\rho(x) = M'(x) =$ linear density at x.

Current

- $Q(t) =$ the charge that has passed a point in a circuit up to time t.
- $I(t) = Q'(t) =$ current at that point at time t.

Marginal Cost

- $C(x) =$ cost of producing x units of a commodity.
- $MC(x) = C'(x) =$ marginal cost when x units have been produced.
- $MC(x) \approx$ the cost of producing an additional unit if x units have already been produced (in other words, the cost of the "$(x+1)$st" unit).

Section 3.8: Related Rates

A Balloon

Problem. Air is blown into a balloon at a rate of 15 cm^3 per second. At what rate is the radius increasing when the volume is 400 cm^3?

Running on a Baseball Field

Problem. A baseball diamond is a square with 90 ft sides. In a game of baseball, Alex is located on 1st base, and Bob is located on second base. When the batter hits the ball, Alex starts running towards second base at 20 ft/s, and Bob starts running towards third base at 25 ft/s. How fast is the distance between Alex and Bob changing when Alex is halfway between first and second base? Is the distance increasing or decreasing?

Conical Water Tank

Problem. Water is pumped into a conical water tank at a rate of 0.5 cubic feet per second. The tank is 20 feet deep, and the diameter at the top is 10 feet. Find the rate at which the water level is rising when the water is 8 feet deep.

Problem. The length of a rectangle is increasing at a rate of 8 cm/s, and its width is increasing at a rate of 6 cm/s. If the length is 5 cm and the width is 4 cm, how fast is the area of the rectangle increasing?

Problem. A rectangular tank with a 5 meter by 8 meter base is filled with water at a rate of 120 m^3/min. How fast is the depth of the water increasing?

Problem. A street light is mounted at the top of a 20 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 4 ft/s along a straight path. How fast is the tip of his shadow moving when he is 50 ft from the pole?

How fast is the length of his shadow increasing when he is 50 ft from the pole?

Ships

Problem. At 12:00 p.m., ship A is located 100 miles west of ship B. Ship A is traveling south at 50 mi/h, and ship B is traveling north at 60 mi/h. How fast is the distance between the two ships increasing at 2:00?

Motion of a Particle

Problem. A particle in the *xy*-plane follows the path $y = (4x - 3)^2$. As it reaches the point $(2, 25)$, its *y*-coordinate is increasing at a rate of 5 units per second. How fast is its *x*-coordinate increasing at this point?

Water Trough

Problem. A trough is 12 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 15 ft^3/min, how fast is the water level rising when the water is 8 inches deep?

Problem. A television camera is positioned 5000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Assume the speed of the rocket is 1000 ft/s when it has risen 12000 ft. How fast is the camera's angle of elevation changing when the rocket has risen 12000 ft.?

Section 3.9: Linear Approximations and Differentials

Linearizations

Definition (Informal)

Suppose f is a function defined near a number a. The *linearization* of f at a is a linear function L that provides a good approximation to f for values of x near a. L is also called the *linear approximation* to f near a.

Definition

Suppose f is a function that is differentiable at a. Then the linearization of f at a is the function L whose graph is the tangent line to f at a.

Example and Equation for Linearizations

Problem. Find the linearization of $f(x) = x^2$ near 3. Use the linearization to estimate 3.05^2, and compare this estimate to the true value of 3.05^2 using a calculator.

Linearization Equation. Assume f is differentiable at a. The linearization of f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

Differentials

Definition

Assume $y = f(x)$.

- The *differential* of x is the variable dx.
- dx can be any number, and it represents a change in x (usually a small change).
- The *differential* of y is the variable dy defined by

$$dy = f'(x)dx.$$

Geometric Meaning of Differentials

- We know $\Delta x =$ any change in x.
- dx is also any change in x, so

$$dx = \Delta x.$$

- Δy is the **actual** change in y corresponding to Δx, so

$$\Delta y = f(x + \Delta x) - f(x).$$

- dy is the change in y corresponding to dx, calculated using the **linear approximation** instead of the function f, so

$$dy = f'(x)dx.$$